Asymptotic enumeration of Eulerian orientations for graphs with strong mixing properties

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Abstract

We prove an asymptotic formula for the number of Eulerian orientations for graphs with strong mixing properties and with vertices having even degrees. The exact value is determined up to the multiplicative error $O(n^{-1+\varepsilon})$, where n is the number of vertices.

1 Introduction

An Eulerian orientation of the graph G is an orientation of its edges such that for every vertex the number of incoming edges and outgoing edges are equal. We denote $EO(G)$ the number of Eulerian orientations of graph G. It is easy clear that $EO(G) = 0$, if the degree of at least one vertex of G is odd. An Eulerian orientation of the complete graph K_n is called regular tournament.

The problem of counting the number of Eulerian orientations of an undirected simple graph (i.e. graph without loops and multiple edges) is complete for the class $\#P$. Thus this problem is difficult in terms of complexity theory. We recall also that this problem can be reduced to counting perfect matching for some class of bipartite graphs for which it can be done approximately with high probability in polynomial time, see [8].

In addition, for the case of loopless $2d$-regular graph G with n vertices the following estimates hold, see [6], [12]:

$$2^d \left(\frac{(2d-1)!!}{d!}\right)^{n-1} \leq EO(G) \leq \left(\frac{(2d)!}{d! \cdot d!}\right)^{n/2}. \tag{1.1}$$

An improvement of the upper bound for the regular graph case and some additional studies in this direction were fulfilled in [7].

Even for a complete graph K_n the exact expression of the number of Eulerian orientations is unknown and only the asymptotic formula was obtained (see [10]): for odd $n \to \infty$

$$EO(K_n) = \left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2} n^{1/2} e^{-1/2} \left(1 + O(n^{-1/2+\varepsilon})\right) \tag{1.2}$$

for any fixed $\varepsilon > 0$.

In [2] the approach of [10] was generalized. In particular, the asymptotic behaviour of the number of Eulerian orienations was determined for graphs with large algebraic connectivity. This class of graphs we mean as the class of graphs having strong mixing properties and it can be also defined as the class of graphs having large Cheeger constant (isoperimetric number) or sufficiently large spectral gap from 1 for the second largest eigenvalue of the transition probability matrix of the random walk on the graph and large degree of each vertex. The equivalence of these definitions was proved, for example, in [3].

In the present work we continue studies of [2], [3], [10]. We prove the asymptotic formula for the number of Eulerian orientations of graphs having strong mixing properties. This result is presented in detail in Section 2 of the present work and was given for the first time, but without proof, in [3].

Actually, the estimation of the number of Eulerian orientations was reduced in [2] to estimating of an n-dimensional integral which is close to Gaussian-type. In Sections 3, 5, 6 we develop an approach for estimating of integrals of such a type.

We note also that in [1] the asymptotic behaviour of the number of Eulerian circuits was determined for graphs having strong mixing properties. Apparently, proceeding from the results of [1] and the estimates of the present work, it is possible to prove the asymptotic formula for the number of Eulerian circuits given in [3]. In a subsequent paper we plan to develop this approach.

2 Main result

Let G be an undirected simple graph with vertex set VG and edge set EG. We define $n \times n$ matrix Q by

$$Q_{jk} = \begin{cases} -1, & \{v_j, v_k\} \in EG, \\ d_j, & j = k, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $n = |VG|$ and d_j denotes the degree of $v_j \in VG$. The matrix $Q = Q(G)$ is called the Laplacian matrix of the graph G. The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of the matrix Q are always non-negative real numbers and $\lambda_1 = 0$. The eigenvalue $\lambda_2 = \lambda_2(G)$ is called the algebraic connectivity of the graph G. In addition the following inequalities hold:

$$2 \min_j d_j - n + 2 \leq \lambda_2 \leq \frac{n}{n-1} \min_j d_j. \tag{2.2}$$

For more information on the spectral properties of graphs see, for example, [4] and [11].

According to the Kirchhoff Matrix-Tree-Theorem, see [5], we have that

$$t(G) = \frac{1}{n} \lambda_2 \lambda_3 \cdots \lambda_n = \det M_{11}, \tag{2.3}$$

where $t(G)$ denotes the number of spanning trees of the graph G and M_{11} results from deleting the first row and the first column of Q.

We call the graph G as γ-mixing graph, $\gamma > 0$, if

$$\text{the algebraic connectivity } \lambda_2 = \lambda_2(G) \geq \gamma|VG|. \tag{2.4}$$

In addition, we recall that (see, [3]):

- The class of γ-mixing graphs can be also defined as the class of graphs having sufficiently large Cheeger constant (isoperimetric number) or spectral gap from 1 for the second largest eigenvalue of the transition probability matrix of the random walk on the graph and sufficiently large degree of each vertex.
- Almost all graphs (asymptotically, in some probabilistic sense) are γ-mixing.

The main result of the present work is the following theorem.

Theorem 2.1. *Let G be an undirected simple graph with n vertices $v_1, v_2, \ldots, v_n$ having even degrees. Let G satisfy (2.4) for some $\gamma > 0$. Then*

$$\begin{aligned} EO(G) &= (1+\delta(G))\left(2^{|EG|+\frac{n-1}{2}}\pi^{-\frac{n-1}{2}}\frac{1}{\sqrt{t(G)}}\prod_{\{v_j,v_k\}\in EG} P_{jk}\right), \\ P_{jk} &= 1 - \frac{1}{4(d_j+1)^2} - \frac{1}{2(d_j+1)(d_k+1)} - \frac{1}{4(d_k+1)^2}, \end{aligned} \tag{2.5}$$

where d_j denotes the degree of vertex v_j, $t(G)$ denotes the number of spanning trees of the graph G and for any $\varepsilon > 0$

$$|\delta(G)| \leq Cn^{-1+\varepsilon}, \tag{2.6}$$

where constant $C > 0$ depends only on γ and ε.

Proof of Theorem 2.1 is given in Section 4. This proof is based on results presented in Section 3.

Remark 2.1. *For the complete graph $\lambda_2(K_n) = n$, $EK_n = \frac{n(n-1)}{2}$, $t(K_n) = n^{n-2}$,*

$$\begin{aligned} \prod_{\{v_j,v_k\}\in EK_n} P_{jk} &= \left(1 - \frac{1}{4n^2} - \frac{1}{2n^2} - \frac{1}{4n^2}\right)^{\frac{n(n-1)}{2}} = \\ &= \left(e^{\ln\left(1-\frac{1}{n^2}\right)}\right)^{\frac{n(n-1)}{2}} = e^{-1/2} + O(n^{-1}). \end{aligned} \tag{2.7}$$

The result of Theorem 2.1 for this case improve the error estimate in (1.2).

3 Asymptotic estimates of an integral

We fix constants $a, b, \varepsilon > 0$. In this section we use notation $f = O(g)$ meaning that $|f| \le c|g|$ for some $c > 0$ depending only on a, b and ε.

Let $p \ge 1$ be a real number and $\vec{x} \in \mathbb{R}^n$. Let

$$\|\vec{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}. \tag{3.1}$$

For $p = \infty$ we have the maximum norm

$$\|\vec{x}\|_\infty = \max_j |x_j|. \tag{3.2}$$

The matrix norm corresponding to the p-norm for vectors is

$$\|A\|_p = \sup_{\vec{x} \ne 0} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p}. \tag{3.3}$$

One can show that for symmetric matrix A and $p \ge 1$

$$\|A\|_p \ge \|A\|_2 . \tag{3.4}$$

For invertible matrices one can define the condition number

$$\mu_p(A) = \|A\|_p \cdot \left\|A^{-1}\right\|_p \ge \left\|AA^{-1}\right\|_p = 1. \tag{3.5}$$

Let I be identity $n \times n$ matrix and $A = I + X$ be such a matrix that:

$$\begin{gathered} A \text{ is positive definite symmetric matrix,} \\ |X_{jk}| \le a/n, \quad X_{jj} = 0, \quad \|A^{-1}\|_2 \le b. \end{gathered} \tag{3.6}$$

Note that

$$\|A^{-1}\|_2^{-1} \le \|A\|_2 \le \|A\|_\infty = \|A\|_1 = \max_j \sum_{k=1}^{n} |A_{jk}| = O(1). \tag{3.7}$$

We recall that (see Lemma 3.2 of [2]), under assumptions (3.6),

$$\mu_\infty(A) = \mu_1(A) = O(\mu_2(A)). \tag{3.8}$$

Using (3.4), (3.6), (3.7) and (3.8), it is not difficult to prove the following proposition.

Proposition 3.1. *Let A satisfy (3.6). Then*

$$\|A^{-1}\|_\infty = \|A^{-1}\|_1 = O(1), \tag{3.9}$$

$$|X'_{jk}| = O(n^{-1}), \tag{3.10}$$

where

$$X' = A^{-1} - I = A^{-1}(I - A) = -A^{-1}X. \tag{3.11}$$

We use the following notation:

$$< g >_{F,\Omega}= \int\limits_{\Omega} g(\vec{\theta})e^{F(\vec{\theta})}d\vec{\theta}, \tag{3.12}$$

For $r > 0$ let

$$< g >_{F,r}=< g >_{F,U_n(rn^{\varepsilon})}, \tag{3.13}$$

where

$$U_n(\rho) = \{(\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n \ : \ |\theta_j| \le \rho \text{ for all } j = 1 \dots n\}. \tag{3.14}$$

Let

$$F(\vec{\theta}) = -\vec{\theta}^T A\vec{\theta} + R(\vec{\theta}), \tag{3.15}$$

where A satisfy (3.6). We consider the following assumptions on function R:

$$R(\vec{\theta}) \le c_1 \frac{\vec{\theta}^T A\vec{\theta}}{n}, \tag{3.16}$$

$$\left\| \frac{\partial R(\vec{\theta})}{\partial \vec{\theta}} \right\|_{\infty} \le c_2 \frac{\|\vec{\theta}\|_{\infty}^3}{n} \tag{3.17}$$

For $R \equiv 0$ we denote $< g >_{\Omega}=< g >_{F,\Omega}$, $< g >_r=< g >_{F,r}$, $< g >=< g >_{+\infty}$.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be such that $U_n(r_1 n^{\varepsilon}) \subset \Omega \subset U_n(r_2 n^{\varepsilon})$ for some $r_2 > r_1 > 0$. Let A satisfy (3.6) and assumptions (3.16), (3.17) hold for some $c_1, c_2 > 0$. Then*

$$< 1 >_{\Omega}= \left(1 + O\left(\exp(-c_3 n^{2\varepsilon})\right)\right) < 1 >, \tag{3.18}$$

$$< 1 >_{F,\Omega}= O\left(< 1 >\right), \tag{3.19}$$

$$< \theta_k^4 >_{F,\Omega}= \frac{3}{4} < 1 >_{F,\Omega} +O(n^{-1+7\varepsilon}) < 1 > \tag{3.20}$$

and, for $k \neq l$:

$$< \theta_k \theta_l^3 >_{F,\Omega}= O(n^{-1+7\varepsilon}) < 1 >, \tag{3.21}$$

$$< \theta_k^2 \theta_l^2 >_{F,\Omega}= \frac{1}{4} < 1 >_{F,\Omega} +O(n^{-1+7\varepsilon}) < 1 >, \tag{3.22}$$

where F is defined by (3.15) and $c_3 = c_3(r_1, r_2, c_1, c_2, a, b, \varepsilon) > 0$.

Proof of Proposition 3.2 is given in Section 5.

4 Proof of Theorem 2.1

The Laplacian matrix Q of the graph G has the eigenvector $[1,1,\ldots,1]^T$, corresponding to the eigenvalue $\lambda_1 = 0$. Let $\hat{Q} = Q + J$, where J denotes the matrix with every entry 1. Note that Q and $\hat{Q}$ have the same set of eigenvectors and eigenvalues, except for the eigenvalue corresponding to the eigenvector $[1,1,\ldots,1]^T$, which equals 0 for Q and n for $\hat{Q}$. Using (2.3), we find that

$$t(G) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n = \frac{\det\hat{Q}}{n^2}. \tag{4.1}$$

Using (3.4), we get that

$$\lambda_n = ||Q||_2 \leq ||\hat{Q}||_2 \leq ||\hat{Q}||_1 = \max_j \sum_{k=1}^{n} |\hat{Q}_{jk}| = n. \tag{4.2}$$

Then, we find that

$$||\hat{Q}^{-1}||_2 = \frac{1}{\lambda_2} \leq \frac{1}{\gamma n}. \tag{4.3}$$

Using (2.2), we get that

$$n-1 \geq d_j \geq \lambda_2 \frac{n-1}{n} \geq \gamma(n-1). \tag{4.4}$$

We recall that under assumtions of Theorem 2.1 (see, Proposition 6.1 of [2])

$$EO(G) = \left(1 + O\left(n^{-1+6\varepsilon}\right)\right) 2^{|EG|-1/2}\pi^{-n+1/2} n\,\mathrm{Int}, \tag{4.5}$$

where

$$\begin{gathered}\mathrm{Int} = \int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-\frac{1}{2}\,\vec{\xi}^T\hat{Q}\vec{\xi} - \frac{1}{12}\sum_{\{v_j,v_k\}\in EG} \Delta_{jk}^4\right) d\vec{\xi}, \\ \Delta_{jk} = \xi_j - \xi_k.\end{gathered} \tag{4.6}$$

Let

$$\theta_k = \sqrt{(d_k+1)/2}\,\xi_k. \tag{4.7}$$

Then

$$\begin{aligned}\Delta_{jk}^4 = \frac{4\theta_j^4}{(d_j+1)^2} - 4\frac{4\theta_j^3\theta_k}{(d_j+1)^{3/2}(d_k+1)^{1/2}} + 6\frac{4\theta_j^2\theta_k^2}{(d_j+1)(d_k+1)} - \\ -4\frac{4\theta_k^3\theta_j}{(d_k+1)^{3/2}(d_j+1)^{1/2}} + \frac{4\theta_k^4}{(d_k+1)^2}.\end{aligned} \tag{4.8}$$

$$-\frac{1}{2}\,\vec{\xi}^T\hat{Q}\vec{\xi} = -\vec{\theta}^T A\vec{\theta}, \tag{4.9}$$

where

$$A_{jk} = \frac{1}{\sqrt{(d_j+1)(d_k+1)}}\hat{Q}_{jk}. \tag{4.10}$$

For a subset Θ of EG let

$$R_\Theta(\vec{\theta}) = -\frac{1}{12}\sum_{\{v_j,v_k\}\in\Theta\subset EG}\Delta_{jk}^4. \tag{4.11}$$

Using (4.3), (4.4), (4.8) it is not difficult to find that the assumptions of Proposition 3.2 for

$$\Omega = \{\vec{\theta}\in\mathbb{R}^n : \vec{\xi}\in U_n(n^{-1/2+\varepsilon})\} \tag{4.12}$$

and

$$F(\vec{\theta}) = F_\Theta(\vec{\theta}) = -\vec{\theta}^T A\vec{\theta} + R_\Theta(\vec{\theta}) \tag{4.13}$$

hold with constants r_1, r_2, c_1, c_2, a, b depending only on γ.

Using (3.20), (3.21), (3.22) and (4.8), we get that

$$< \Delta_{jk}^4 >_{F_\Theta,\Omega} = \left(\frac{3}{4}\frac{4}{(d_j+1)^2} + \frac{6}{4}\frac{4}{(d_j+1)(d_k+1)} + \frac{3}{4}\frac{4}{(d_k+1)^2}\right) < 1 >_{F_\Theta,\Omega} + \\ +O(n^{-3+7\varepsilon}) < 1 > . \tag{4.14}$$

Using (3.19), we get that

$$< e^{-\frac{1}{12}\Delta_{jk}^4} >_{F_\Theta,\Omega} = P_{jk} < 1 >_{F_\Theta,\Omega} + O(n^{-3+7\varepsilon}) < 1 >, \tag{4.15}$$

where

$$P_{jk} = 1 - \frac{1}{4(d_j+1)^2} - \frac{1}{2(d_j+1)(d_k+1)} - \frac{1}{4(d_k+1)^2}. \tag{4.16}$$

Note that

$$1 - \frac{1}{n^2} \le P_{jk} \le 1. \tag{4.17}$$

Using (3.19), (4.15), we can gradually remove all the edges in $R_{EG}(\vec{\theta})$ and obtain that

$$< 1 >_{F_{EG},\Omega} = \prod_{\{v_j,v_k\}\in EG} P_{jk} < 1 >_\Omega + O(n^{-1+7\varepsilon}) < 1 > . \tag{4.18}$$

Combining (3.18) and (4.18), we get that

$$< 1 >_{F_{EG},\Omega} = \left(\prod_{\{v_j,v_k\}\in EG} P_{jk} + O(n^{-1+7\varepsilon})\right) < 1 > . \tag{4.19}$$

Then

$$\begin{aligned}
\mathrm{Int} = \int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-\frac{1}{2}\vec{\xi}^T\hat{Q}\vec{\xi} - \frac{1}{12}\sum_{\{v_j,v_k\}\in EG}\Delta_{jk}^4\right)d\vec{\xi} = \\
= \int\limits_{U_n(n^{-1/2+\varepsilon})} \exp\left(-\vec{\theta}^T A\vec{\theta} + R_{EG}(\vec{\theta})\right)d\vec{\xi} = \\
= \left(\prod_{(v_j,v_k)\in EG} P_{jk} + O(n^{-1+7\varepsilon})\right)\int\limits_{\mathbb{R}^n}\exp\left(-\vec{\theta}^T A\vec{\theta}\right)d\vec{\xi} = \\
\left(1+O(n^{-1+7\varepsilon})\right)\frac{(2\pi)^{n/2}}{\sqrt{\det\hat{Q}}}\prod_{\{v_j,v_k\}\in EG} P_{jk}.
\end{aligned} \tag{4.20}$$

Combining (4.1), (4.5), (4.20), we obtain (2.5).

5 Proof of Proposition 3.2

Let

$$\vec{\phi}(\vec{\theta}) = (\phi_1(\vec{\theta}), \phi_2(\vec{\theta}), \ldots, \phi_n(\vec{\theta})) = A\vec{\theta}. \tag{5.1}$$

According to (3.6), $A = I + X$, $X_{jj} = 0$, and so for some $g_1(\vec{\theta}) = g_1(\theta_2, \ldots, \theta_n)$

$$\vec{\theta}^T A\vec{\theta} = \phi_1^2(\vec{\theta}) + g_1(\vec{\theta}). \tag{5.2}$$

Using (3.6), (5.2) and estimating insignificant parts of Gaussian integral of the following type:

$$\int (\max\{|x|, k_1\})^s\, e^{-(x-k_2)^2}dx, \tag{5.3}$$

we find that for $r > 0$, $s \geq 0$

$$\begin{aligned}
< \|\vec{\theta}\|_\infty^s > = \int\limits_{\mathbb{R}^n}\|\vec{\theta}\|_\infty^s e^{-\vec{\theta}^T A\vec{\theta}}d\vec{\theta} = \\
= \int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty} e^{-\,g_1(\theta_2,\ldots,\theta_n)}\left(\int\limits_{-\infty}^{+\infty}\|\vec{\theta}\|_\infty^s e^{-\phi_1(\vec{\theta})^2}d\theta_1\right)d\theta_2\ldots d\theta_n \\
= \left(1 + O\left(\exp(-c_4 n^{2\varepsilon})\right)\right)\int\limits_{|\phi_1(\vec{\theta})|\leq rn^\varepsilon}\|\vec{\theta}\|_\infty^s e^{-\,\vec{\theta}^T A\vec{\theta}}d\vec{\theta},
\end{aligned} \tag{5.4}$$

where $c_4 = c_4(r, \varepsilon, s) > 0$. Combining similar expressions for $\phi_1, \phi_2, \ldots \phi_n$, we get that

$$\int\limits_{||\vec{\phi}(\vec{\theta})||_\infty\leq rn^\varepsilon}\|\vec{\theta}\|_\infty^s e^{-\vec{\theta}^T A\vec{\theta}}d\vec{\theta} = \left(1 + O\left(\exp(-c_5 n^{2\varepsilon})\right)\right) < \|\vec{\theta}\|_\infty^s >, \tag{5.5}$$

where $c_5 = c_5(r, \varepsilon, s) > 0$. Combining (3.9), (5.1) and (5.5) with $s = 0$, we obtain (3.18).

Using (3.16), we find that

$$| < 1 >_{F,\Omega} | \leq \int\limits_{\Omega} |e^{F(\vec{\theta})}| d\vec{\theta} \leq \int\limits_{\mathbb{R}^n} e^{-\vec{\theta}^T A \vec{\theta} + \frac{c_1}{n} \vec{\theta}^T A \vec{\theta}} d\vec{\theta} = O(< 1 >). \tag{5.6}$$

In order to prove (3.20) - (3.22) we use the following two lemmas. The proofs of them are given in Section 6.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be such that $U_n(r_1 n^{\varepsilon}) \subset \Omega \subset U_n(r_2 n^{\varepsilon})$ for some $r_2 > r_1 > 0$. Let A satisfy (3.6) and assumptions (3.16) and (3.17) hold. Let $P = P(x) = O(|x|^s)$ for some fixed $s \geq 0$. Then for any $T(\vec{\theta})$, $|T(\vec{\theta})| \leq P(\|\vec{\theta}\|_\infty)$:*

$$< T(\vec{\theta}) >_{\mathbb{R}^n \setminus \Omega} = O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \tag{5.7}$$

and for any $\tilde{T}(\vec{\theta}) = \tilde{T}(\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_n)$, $\tilde{T}(\vec{\theta}) \leq P(\|\vec{\theta}\|_\infty)$:

$$\begin{aligned} < \phi_k^2(\vec{\theta}) \tilde{T}(\vec{\theta}) >_{F,\Omega} = \frac{1}{2} < \tilde{T}(\vec{\theta}) >_{F,\Omega} + O(n^{-1+4\varepsilon}) < |\tilde{T}(\vec{\theta})| >_{F,\Omega} + \\ + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \end{aligned} \tag{5.8}$$

$$\begin{aligned} < \phi_k^4(\vec{\theta}) \tilde{T}(\vec{\theta}) >_{F,\Omega} = \frac{3}{4} < \tilde{T}(\vec{\theta}) >_{F,\Omega} + O(n^{-1+4\varepsilon}) < |\tilde{T}(\vec{\theta})| >_{F,\Omega} + \\ + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \end{aligned} \tag{5.9}$$

$$\begin{aligned} < \phi_k(\vec{\theta}) \tilde{T}(\vec{\theta}) >_{F,\Omega} = O(n^{-1+4\varepsilon}) < |\tilde{T}(\vec{\theta})| >_{F,\Omega} + \\ + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \end{aligned} \tag{5.10}$$

$$\begin{aligned} < \phi_k^3(\vec{\theta}) \tilde{T}(\vec{\theta}) >_{F,\Omega} = O(n^{-1+6\varepsilon}) < |\tilde{T}(\vec{\theta})| >_{F,\Omega} + \\ + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \end{aligned} \tag{5.11}$$

where function F is defined by (3.15), vector $\vec{\phi}(\vec{\theta})$ is defined by (5.1) and constant $c_6 = c_6(r_1, r_2, c_1, c_2, a, b, \varepsilon, P) > 0$.

Lemma 5.2. *Let assumptions of Lemma 5.1 hold. Let $s_1, s_2, \ldots, s_n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned} M(\vec{x}) = x_1^{s_1} \cdots x_n^{s_n}, \\ s = s_1 + \ldots + s_n > 0. \end{aligned} \tag{5.12}$$

Let

$$s_k = 0 \textit{ and } |\{j : s_j \neq 0\}| \leq 3. \tag{5.13}$$

Then

$$< \phi_k(\vec{\theta}) M(\vec{\phi}(\vec{\theta})) >_{F,\Omega} = O(s n^{-1+(s+4)\varepsilon}) < 1 > . \tag{5.14}$$

Using (3.10), we find that

$$\theta_k = \phi_k + \vec{\alpha}^T\vec{\phi}, \quad \|\vec{\alpha}\|_\infty = O(n^{-1}). \tag{5.15}$$

Combining (5.6), Lemma 5.1 and Lemma 5.2, we obtain that:

$$\begin{aligned} < \delta_k(\vec{\theta})^2 >_{F,\Omega} = O(n^{-2}) \left(\sum_j < \phi_j(\vec{\theta})^2 >_{F,\Omega} + \sum_{j_1 \neq j_2} | < \phi_{j_1}(\vec{\theta})\phi_{j_2}(\vec{\theta}) >_{F,\Omega} | \right) = \\ = \left(O(n^{-1}) + O(n^{-1+5\varepsilon}) \right) < 1 > = O(n^{-1+5\varepsilon}) < 1 >, \end{aligned} \tag{5.16}$$

$$\begin{aligned} < \delta_k(\vec{\theta})^4 >_{F,\Omega} &= O(n^{-4}) \sum_j < \phi_j(\vec{\theta})^4 >_{F,\Omega} + \\ &+ O(n^{-4}) \sum_{j_1 \neq j_2} < \phi_{j_1}(\vec{\theta})^2 \phi_{j_2}(\vec{\theta})^2 >_{F,\Omega} + \\ &+ O(n^{-4}) \sum_{j_1 \neq j_2} | < \phi_{j_1}(\vec{\theta}) \phi_{j_2}(\vec{\theta})^3 >_{F,\Omega} | + \\ &+ O(n^{-4}) \sum_{j_1 \neq j_2 \neq j_3} | < \phi_{j_1}(\vec{\theta}) \phi_{j_2}(\vec{\theta}) \phi_{j_3}(\vec{\theta})^2 >_{F,\Omega} | + \\ &+ O(n^{-4}) \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} | < \phi_{j_1}(\vec{\theta}) \phi_{j_2}(\vec{\theta}) \phi_{j_3}(\vec{\theta}) \phi_{j_4}(\vec{\theta}) >_{F,\Omega} | = \\ = (O(n^{-3}) + O(n^{-2+4\varepsilon}) &+ O(n^{-3+7\varepsilon}) + O(n^{-2+7\varepsilon}) + O(n^{-1+7\varepsilon})) < 1 > = \\ &= O(n^{-1+7\varepsilon}) < 1 >, \end{aligned} \tag{5.17}$$

where

$$\delta_k(\vec{\theta}) = \theta_k - \phi_k(\vec{\theta}). \tag{5.18}$$

According to (3.6), we have that

$$\delta_k(\vec{\theta}) = \delta_k(\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_n) \tag{5.19}$$

Using (5.6), (5.16), (5.17), (5.19) and Lemma 5.1, we obtain that:

$$\begin{aligned} < \phi_k(\vec{\theta})\delta_k(\vec{\theta})^3 >_{F,\Omega} = \; O(n^{-1+4\varepsilon}) < |\delta_k(\vec{\theta})|^3 >_{F,\Omega} + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 > = \\ = O(n^{-1+7\epsilon}) < 1 >, \end{aligned} \tag{5.20}$$

$$\begin{aligned} < \phi_k(\vec{\theta})^2\delta_k(\vec{\theta})^2 >_{F,\Omega} = \; \left(\frac{1}{2} + O(n^{-1+4\varepsilon}) \right) < \delta_k(\vec{\theta})^2 >_{F,\Omega} + \\ + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 > = \\ = O(n^{-1+5\epsilon}) < 1 >, \end{aligned} \tag{5.21}$$

$$\begin{aligned} < \phi_k(\vec{\theta})^3\delta_k(\vec{\theta}) >_{F,\Omega} = \; O(n^{-1+6\varepsilon}) < |\delta_k(\vec{\theta})| >_{F,\Omega} + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 > = \\ = O(n^{-1+7\epsilon}) < 1 >, \end{aligned} \tag{5.22}$$

$$< \theta_k^4 >_{F,\Omega} = < (\phi_k(\vec{\theta}) + \delta_k(\vec{\theta}))^4 >_{F,\Omega} = < \phi_k(\vec{\theta})^4 >_{F,\Omega} + O(n^{-1+7\epsilon}) < 1 > = \\ = \frac{3}{4} < 1 >_{F,\Omega} + O(n^{-1+7\epsilon}) < 1 > . \tag{5.23}$$

In a similar way as in (5.16), (5.17), we find that

$$< \delta_k(\vec{\theta})^2 \theta_l^2 >_{F,\Omega} = O(n^{-2}) \sum_j < \phi_j(\vec{\theta})^2 \theta_l^2 >_{F,\Omega} + \\ + O(n^{-2}) \sum_{j_1 \neq j_2} | < \phi_{j_1}(\vec{\theta}) \phi_{j_2}(\vec{\theta}) \theta_l^2 >_{F,\Omega} | = \\ = \big(O(n^{-1+2\varepsilon}) + O(n^{-1+7\varepsilon}) \big) < 1 > = O(n^{-1+7\varepsilon}) < 1 >, \tag{5.24}$$

$$< \delta_k(\vec{\theta}) \theta_l^3 >_{F,\Omega} = O(n^{-1}) \sum_{j \neq l} < \phi_j(\vec{\theta}) \theta_l^3 >_{F,\Omega} + O(n^{-1}) < \phi_l(\vec{\theta}) \theta_l^3 >_{F,\Omega} = \\ = \big(O(n^{-1+7\varepsilon}) + O(n^{-1+4\varepsilon}) \big) < 1 > = O(n^{-1+7\varepsilon}) < 1 > . \tag{5.25}$$

Using (5.6), (5.19), (5.24), (5.25) and Lemma 5.1, we obtain that:

$$< \phi_k(\vec{\theta}) \delta_k(\vec{\theta}) \theta_l^2 >_{F,\Omega} = O(n^{-1+4\varepsilon}) < |\delta_k(\vec{\theta}) \theta_l^2| >_{F,\Omega} + O\left(\exp(-c_6 n^{2\varepsilon}) \right) < 1 > = \\ = O(n^{-1+7\epsilon}) < 1 >, \tag{5.26}$$

$$< \theta_k \theta_l^3 >_{F,\Omega} = < (\phi_k(\vec{\theta}) + \delta_k(\vec{\theta})) \theta_l^3 >_{F,\Omega} = O(n^{-1+7\epsilon}) < 1 >, \tag{5.27}$$

$$< \theta_k^2 \theta_l^2 >_{F,\Omega} = < (\phi_k(\vec{\theta}) + \delta_k(\vec{\theta}))^2 \theta_l^2 >_{F,\Omega} = < \phi_k(\vec{\theta})^2 \theta_l^2 >_{F,\Omega} + O(n^{-1+7\epsilon}) < 1 > = \\ = \frac{1}{2} < \phi_l(\vec{\theta}) + \delta_l(\vec{\theta}))^2 >_{F,\Omega} + O(n^{-1+7\epsilon}) < 1 > = \\ = \frac{1}{4} < 1 >_{F,\Omega} + O(n^{-1+7\epsilon}) < 1 > . \tag{5.28}$$

6 Proofs of Lemma 5.1 and Lemma 5.2

Let

$$\vec{\theta}^{(k)} = (\theta_1, \ldots, \theta_{k-1}, 0, \theta_{k+1}, \ldots, \theta_n). \tag{6.1}$$

Proof of Lemma 5.1. Using (3.9), (5.1) and (5.5), we find that

$$| < T >_{\mathbb{R}^n \setminus \Omega} | \leq \int\limits_{\mathbb{R}^n \setminus \Omega} P(\|\vec{\theta}\|_\infty^s) e^{-\vec{\theta}^T A \vec{\theta}} d\vec{\theta} = O\left(\exp(-c_6 n^{2\varepsilon}) \right) < 1 > . \tag{6.2}$$

For simplicity, let $k = 1$. Using (6.2), we get that

$$< \phi_1^p(\vec{\theta})\tilde{T}(\vec{\theta}) >_{F,\Omega} = \int\limits_{\Omega} \phi_1^p(\vec{\theta})\tilde{T}(\theta_2, \ldots, \theta_n) e^{-\vec{\theta}^T A\vec{\theta} + R(\vec{\theta})} d\vec{\theta} =$$

$$= \int\limits_{U_n(r_2 n^{\varepsilon})} \phi_1^p(\vec{\theta})\tilde{T}(\theta_2, \ldots, \theta_n) e^{-\vec{\theta}^T A\vec{\theta} + R(\vec{\theta})} d\vec{\theta} + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 >, \quad p = 1, 2, 3, 4. \tag{6.3}$$

Using (3.17), find that for $\vec{\theta} \in U_n(r_2 n^{\varepsilon})$

$$R(\vec{\theta}) - R(\vec{\theta}^{(1)}) = O(n^{-1+4\varepsilon}). \tag{6.4}$$

Using (5.2), we get that

$$\int\limits_{U_n(r_2 n^{\varepsilon})} \phi_1^p(\vec{\theta})\tilde{T}(\theta_2, \ldots, \theta_n) e^{-\vec{\theta}^T A\vec{\theta} + R(\vec{\theta})} d\vec{\theta} + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 > =$$

$$= \int\limits_{-r_2 n^{\varepsilon}}^{r_2 n^{\varepsilon}} \cdots \int\limits_{-r_2 n^{\varepsilon}}^{r_2 n^{\varepsilon}} \tilde{T} e^{-g_1(\theta_2, \ldots, \theta_n) + R(\vec{\theta}^{(1)})} \left(\int\limits_{-r_2 n^{\varepsilon}}^{r_2 n^{\varepsilon}} \phi_1^p e^{-\phi_1^2(\vec{\theta}) + R(\vec{\theta}) - R(\vec{\theta}^{(1)})} d\theta_1 \right) d\theta_2 \ldots d\theta_n, \quad p = 0, 1, 2, 3, 4. \tag{6.5}$$

Combining (3.9) and (6.4), we find that for $\vec{\theta}^{(1)} \in U_n(r_2 n^{\varepsilon})$

$$\int\limits_{-r_2 n^{\varepsilon}}^{r_2 n^{\varepsilon}} \phi_1^p e^{-\phi_1^2(\vec{\theta}) + R(\vec{\theta}) - R(\vec{\theta}^{(1)})} d\theta_1 =$$

$$= \int\limits_{-\infty}^{+\infty} \phi_1^p e^{-\phi_1^2(\vec{\theta})} d\theta_1 + O\left(\exp(-c_7 n^{2\varepsilon})\right) \int\limits_{-\infty}^{+\infty} e^{-\phi_1^2(\vec{\theta})} d\theta_1 + \int\limits_{|\phi_1(\vec{\theta})| \le r_3 n^{\varepsilon}} \phi_1^p e^{-\phi_1^2(\vec{\theta})} \left(e^{R(\vec{\theta}) - R(\vec{\theta}^{(1)})} - 1\right) d\theta_1, \quad p = 0, 1, 2, 3, 4, \tag{6.6}$$

where $c_7 = c_7(r_2, c_1, c_2, a, b, \varepsilon) > 0$, $r_3 = r_3(r_2, c_1, c_2, a, b, \varepsilon) > 0$.

For $p = 2, 4$, we have that:

$$\int_{-\infty}^{+\infty} \phi_1^2 e^{-\phi_1^2(\vec{\theta})} d\theta_1 = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\phi_1^2(\vec{\theta})} d\theta_1,$$
$$\int_{-\infty}^{+\infty} \phi_1^4 e^{-\phi_1^2(\vec{\theta})} d\theta_1 = \frac{3}{4} \int_{-\infty}^{+\infty} e^{-\phi_1^2(\vec{\theta})} d\theta_1, \tag{6.7}$$

$$\int\limits_{|\phi_1(\vec{\theta})|<r_3 n^{\varepsilon}} \phi_1^p e^{-\phi_1^2(\vec{\theta})} \left(e^{R(\vec{\theta})-R(\vec{\theta}^{(1)})} - 1\right) d\theta_1 = O(n^{-1+4\epsilon}) \int_{-\infty}^{+\infty} \phi_1^p e^{-\phi_1^2(\vec{\theta})} d\theta_1 =$$
$$= O(n^{-1+4\epsilon}) \int_{-\infty}^{+\infty} e^{-\phi_1^2(\vec{\theta})} d\theta_1$$
$$\text{for } \vec{\theta}^{(1)} \in U_n(r_2 n^{\varepsilon}), \quad p = 0, 2, 4. \tag{6.8}$$

Combining (6.2)-(6.8), we obtain (5.8) and (5.9).

For $p = 1, 3$, we have that:

$$\int_{-\infty}^{+\infty} \phi_1^p e^{-\phi_1^2(\vec{\theta})} d\theta_1 = 0 \tag{6.9}$$

$$\int\limits_{|\phi_1(\vec{\theta})|\le r_3 n^{\varepsilon}} \phi_1^p e^{-\phi_1^2(\vec{\theta})} \left(e^{R(\vec{\theta})-R(\vec{\theta}^{(1)})} - 1\right) d\theta_1 =$$
$$= \int\limits_{0\le\phi_1(\vec{\theta})\le r_3 n^{\varepsilon}} |\phi_1|^p e^{-\phi_1^2(\vec{\theta})} \left(e^{R(\vec{\theta})-R(\vec{\theta}^{(1)})} - 1\right) d\theta_1 -$$
$$- \int\limits_{r_3 n^{\varepsilon}\le\phi_1(\vec{\theta})\le 0} |\phi_1|^p e^{-\phi_1^2(\vec{\theta})} \left(e^{R(\vec{\theta})-R(\vec{\theta}^{(1)})} - 1\right) d\theta_1 =$$
$$= O(n^{-1+4\epsilon}) \int\limits_{\phi_1\ge 0} |\phi_1|^p e^{-\phi_1^2(\vec{\theta})} d\theta_1 = O(n^{-1+4\epsilon}) \int_{-\infty}^{+\infty} e^{-\phi_1^2(\vec{\theta})} d\theta_1,$$
$$\text{for } \vec{\theta}^{(1)} \in U_n(r_2 n^{\varepsilon}), \quad p = 1, 3. \tag{6.10}$$

Combining (6.2)-(6.6), (6.8), (6.9), (6.10), we obtain (5.8) and (5.9). ■

Proof of Lemma 5.2. Let $T(\vec{\theta})$ satisfy

$$|T(\vec{\theta})| = O(\|\vec{\theta}\|_\infty^s),$$
$$\frac{\partial T(\vec{\theta})}{\partial \theta_k} = O(sn^{-1-\varepsilon}) \sup_{\vec{\theta}\in\Omega} |T(\vec{\theta})|, \quad \vec{\theta} \in \Omega. \tag{6.11}$$

Combining (5.10) and (6.11), we get that

$$\begin{aligned} < \phi_k(\vec{\theta})T(\vec{\theta}) >_{F,\Omega} =&< \phi_k(\vec{\theta})T(\vec{\theta}^{(k)}) >_{F,\Omega} + < \phi_k(\vec{\theta})(T(\vec{\theta}) - T(\vec{\theta}^{(k)}) >_{F,\Omega} = \\ &=< \phi_k(\vec{\theta})T(\vec{\theta}^{(k)}) >_{F,\Omega} + O(sn^{-1-\epsilon}) < \sup_{\vec{\theta}\in\Omega} |\phi_k \theta_k T(\vec{\theta})| >_{F,\Omega} = \\ &= O(sn^{-1+4\epsilon}) < \sup_{\vec{\theta}\in\Omega} |T(\vec{\theta})| >_{F,\Omega} + O\left(\exp(-c_6 n^{2\varepsilon})\right) < 1 > . \end{aligned} \tag{6.12}$$

Using (3.6), we find that for $\vec{\theta} \in \Omega$

$$\frac{\partial M(\vec{\phi}(\vec{\theta}))}{\partial \theta_k} = O(sn^{(s-1)\varepsilon}) \sum_{j:s_j\neq 0} \frac{\partial \vec{\phi}_j(\vec{\theta})}{\partial \theta_k} = O(sn^{-1-\varepsilon}) \sup_{\vec{\theta}\in\Omega} |M(\vec{\phi}(\vec{\theta}))|. \tag{6.13}$$

Combining (5.6) and(6.12), we obtain (5.14) ■

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Warning

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